International Journal of Advanced Mechanics and Control Systems

Volume1, Issue 4 2025

Nonsingular fast terminal sliding-mode control for nonlinear dynamical systems

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Research Article

Date of Submission: 18-08-2025 Date of Acceptance: 15-09-2025 Date of Publication: 18-10-2025

KEY WORDS: terminal sliding mode; fast convergence rate; singularity; Lyapunov stability; nonlinear systems

INTRODUCTION

The sliding-mode control (SMC) approach has been employed extensively in many uncertain systems in recent decades because of its appealing features, including great resilience, order reduction, ease of implementation, and design simplicity. In order to execute SMC design, a switching surface that specifies the required convergence feature must first be defined. Next, a sliding-mode controller that drives the system states to the selected manifold—which is unaffected by uncertainties or disturbances—must be designed [1–5]. Because the linear switching manifolds are often selected with the asymptotical convergence attribute, one feature of the traditional SMC is that the plants typically converge to the equilibrium points asymptotically.

To accomplish finite-time convergence of the system dynamics in the switching surface, terminal SMC has been established in [7–12] using finite-time differential equations [6]. The first-order terminal sliding-mode (TSM) method was first developed for the management of second-order nonlinear systems in the study [7–10]. The TSM method has been extended to high-order single-input and single-output (SISO) linear systems by Yu and Man [11]. To demonstrate the sequential convergence of the sliding variables, a hierarchical TSM structure has been suggested. For multi-input and multi-output (MIMO) linear systems, a generic TSM was then created in [12]. It should be noted that when the system states are far from equilibrium, TSM may not provide the same convergence performance as the traditional sliding-mode approach. A fast terminal sliding-mode (FTSM) idea has been used by Yu and Man [13, 14], ensuring quick transient

convergence at the equilibrium as well as at a distance from it. Nevertheless, it is discovered via control design analysis that the previous TSM and FTSM approaches also have a singularity issue. Several strategies have been put out to get around the problem. Changing the sliding mode from TSM to linear hyperplane-based sliding mode is one method [12]. Another method, known as the "two-phase control strategy," involves moving the trajectory ahead of time to a predetermined area where singularities do not occur [15]. It should be mentioned that in order to prevent singularity, these techniques are using indirect approaches. A novel idea of nonsingular terminal sliding mode (NTSM) has been used in [16, 17] to eliminate the singularity without requiring additional steps. Feng et al. [18] proposed a second-order NTSM deconstructed control technique to linear high-dimensional multivariable systems based on the nonsingular approach. Additionally, certain broader variations of FTSM, NTSM, and TSM have been proposed in [19]. The TSM techniques are very helpful in high-precision control scenarios since they may significantly accelerate the convergence rate. They have so far been used to many different fields, such as robotic manipulators [21, 22] and chaotic systems [20]. However, it should be noted that further research is still required to improve the FTSM method's convergence rate and singularity avoidance.

The finite-time stabilization issue of nonlinear dynamics is studied in this research using the

FTSM approach. In order to demonstrate a quicker rate of convergence than the conventional FTSM, certain additional models are developed, particularly when the system states are far from the equilibrium. The new FTSM proposes an analogous nonsingular fast terminal sliding mode (NFTSM) to eliminate the singularity entirely. The stabilization of a second-order nonlinear system is then thoroughly examined in order to demonstrate the fundamental idea of the NFTSM control approach. The superior qualities of the closed-loop system, including great resilience, quick finite-time convergence, and singularity avoidance, are confirmed. Lastly, numerical simulation on stiff robotic manipulators is shown to verify the suggested method, and the NFTSM technique is proposed to control a class of practical systems, including mechanical systems.

The structure of this document is as follows. In Section 2, we suggest several novel forms of FTSM and examine their pertinent features based on the fundamental notion of TSM. A short summary of some recent findings is also included. The fundamental idea of the NFTSM control approach is introduced in Section 3 by stabilizing a second-order nonlinear system. This technique is subsequently applied to rapid finite-time control of a class of real-world systems. A numerical example supporting the suggested control strategy is carried out in Section 4. The study concludes with some last thoughts in Section 5.

BASIC CONCEPTS ON TSM

In this section, we review the results of TSM in recent studies, propose some new forms of FTSM, and analyze their properties for the application in control of nonlinear systems.

Before we proceed, we consider the following first-order nonlinear dynamics:

$$\dot{x} = -k\,\mathrm{sign}^{a} x, \quad a>0, \ k>0, \qquad (1)$$

where $x \in \mathbb{R}$ and $\mathrm{sign}^{a} x := \mathrm{sign}(x)\,|x|^{a}$. It is clear to see that, given positive odd integers p, q such that $a = p/q$, (1) will reduce to $\dot{x} = -k\,\mathrm{sign}^{a} x = -k x^{a}$, which is the form of conventional TSM. It can be verified that the solution of (1) in the forward time direction is unique for any nonzero initial condition in spite of non-Liptchitz condition [23]. The properties of system (1) are listed: When $a = 1$, the system becomes $\dot{x} = -kx$, and it follows that the system is exponentially stable; When $0<a<1$, compared with the previous case, the system has a lower rate at a distance from zero and converges much more quickly near the origin, which results in finite-time convergence; When $a>1$, it offers a high convergence rate at a distance from zero, and it slows down as it comes close to the origin. Moreover, the more closer, the lower rate, and vice versa.

Definition 1 (*Yu et al.* [*19*])

The TSM and NTSM are described by the following first-order nonlinear differential equations:

$$\sigma(t) = \dot{x} + k\,\mathrm{sign}^{a} x = 0, \quad k>0, \ 0<a<1, \qquad (2a)$$

$$\sigma(t) = x + k'\,\mathrm{sign}^{a'} \dot{x} = 0, \quad k'>0, \ 1<a'<2. \qquad (2b)$$

Remark 1

It is evident that TSM (2a) is equivalent to NTSM (2b) provided that $k' = k^{-a'}$ and $a' = a^{-1}$. Based on the finite-time differential equation [6], we conclude that the dynamics (2a) and (2b) are both globally finite-time stable, and they converge to the equilibrium point within the time

$$T_t = \int_0^{|x(0)|} \frac{1}{kx^{a}}\,dx = \frac{x(0)^{1-a}}{}$$

$k(1-a)$ and $|x(0)|\,k'1/a'$ $\frac{|}{k'1/a'}\cdot|x(0)|^{1-1/a'}$ $\int$

$T_n = {}_x1/a'\ \mathrm{d}x =$ $\overline{\quad}$

$_0$

$\frac{}{1-1/a'}$,

respectively. Note that TSM and NTSM converge in a relatively slow rate when far away from the equilibrium point. The following FTSM and NFTSM concepts are developed to remedy this drawback.

Definition 2

The FTSM is described by the following differential equation:

$$\sigma(t)=x^{\cdot}+k_1\,\mathrm{sign}^{a}1\,x+k_2\,\mathrm{sign}^{a}2\,x=0,\qquad k_1>0,\ \ k_2>0,\ \ a_1\geq 1,\ \ 0<a_2<1. \qquad (3)$$

Remark 2

It can be found that FTSM (3) has the form of FTSM [19] when $a_1=1$, and further, (3)

$=$

$=$

coincides with FTSM in [13, 14] if there exists positive odd integers p, q satisfying a_2 p/q with $p<q$. By solving the differential equation (3), it is concluded that x 0 will be reached in a finite time determined by

$$T_{f1}=\int_0^{|x(0)|}\frac{1}{k_1x^{a1}+k_2x^{a2}}\,\mathrm{d}x=\frac{|x(0)|^{1-a1}}{(1-a_1)}\cdot k^{(1-a1)/a1}\ {}_1$$

$$a_1-a_2\quad a_1-a_2\qquad \cdot\,\mathrm{F}\left(1,\frac{a_1-1}{2a_1-a_2-1};-k2k_{\Gamma}{}^{1}|x(0)|^{a_2-a_1}\right), \qquad (4)$$

where F( ) denotes Gauss' Hypergeometric function [24], and the conditions of a_1, a_2, k_1, k_2 induce that F( ) will keep convergent. Usually, the exact form of F( ) varies with the involved parameters. For example,

$$\mathrm{F}(a,b;b;x)=(1-x)^{-a};\mathrm{F}\left(^{1},1;^{3};-x^{2}\right)=x^{-1}\arctan x. \qquad (5)$$

For more details on Gauss' Hypergeometric function, one can refer to the work [24].

Remark 3

It is noted that FTSM is the combination of cases mentioned at the beginning, which is interpreted in the following: When the system state stays at a distance from equilibrium, $k_1\mathrm{sign}^{a}1\,x$ dominates over $k_2\mathrm{sign}^{a}2\,x$. Thus (3) can be approximated by $\sigma(t)$ x $k_1\mathrm{sign}^{a}1\,x$ 0, which guarantees a high convergence rate; When the system state is close to the origin, the dominant term $k_2\mathrm{sign}^{a}2\,x$ determines finite-time convergence. Hereby, the dynamics converges very quickly in the whole FTSM (3). To demonstrate the fast convergence performance of FTSM, we consider the following sliding modes:

$$\sigma_1=x^{\cdot}+\mathrm{sign}^{0.3}x=0,\qquad \sigma_2=x^{\cdot}+\mathrm{sign}^{3}x+\mathrm{sign}^{0.3}x=0,\qquad \sigma_3=x^{\cdot}+x+\mathrm{sign}^{0.3}x=0 \qquad (6)$$

with the initial state x (0) 5. It can be seen from Figure 1 that FTSMs have better convergence rate than the conventional TSM, and the convergence rate of FTSM at $a_1>1$ is higher than its counterpart at $a_1=1$.

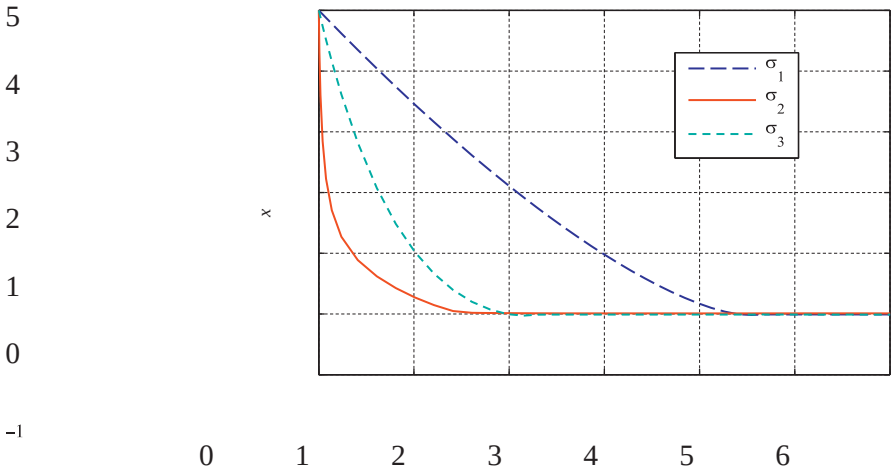


Figure 1. Comparison of TSM, FTSM ($a_1 = 1$) and FTSM ($a_1 > 1$).

Definition 2'

The FTSM can be also given by

$$\sigma(t) = \dot{x} + \bar{k}_1 \operatorname{sign}^{1/\bar{a}_2}(x + \bar{k}_2 \operatorname{sign}^{\bar{a}_1} x) = 0,\ \bar{k}_1 > 0,\ \bar{k}_2 > 0,\ \bar{a}_1 > \bar{a}_2 > 1. \quad (7)$$

Remark 4

It is noted that $|x| \ll |\bar{k}_2 x^{\bar{a}_1}|$ when the state x is far away from zero. At this time, (7) can be approximated by the following equation:

$$\sigma_1(t) = \dot{x} + \bar{k}_1 \bar{k}_2^{-1/\bar{a}_2} \operatorname{sign}^{\bar{a}_1/\bar{a}_2} x = 0,\quad \bar{a}_1/\bar{a}_2 > 1, \quad (8)$$

which prescribes a fast convergence rate. As x closes to zero, x dominates in the term $x + \bar{k}_2 \operatorname{sign}^{\bar{a}_1} x$.

The approximated equation of (7) is given as

$$\sigma_2(t) = \dot{x} + \bar{k}_1 \operatorname{sign}^{1/\bar{a}_2} x = 0,\quad 1/\bar{a}_2 < 1, \quad (9)$$

which is the form of TSM (2a). In terms of finite-time differential equation, it can be easily verified that the dynamics (7) will reach the equilibrium in

$$T_{f2} = \int_{|x(0)|}^{0} \frac{1}{\bar{k}_0 (x + \bar{k}_1 x^{\bar{a}_1})^{1/\bar{a}_2}} dx = \frac{\bar{a}_2 |x(0)|^{1-1/\bar{a}_2}}{\bar{a}_2 - 1} \cdot F\left(\frac{1}{\bar{a}_2 - 1}, \frac{\bar{a}_2 - 1}{\bar{a}_2 - 1}; 1 + \frac{\bar{a}_2 - 1}{(\bar{a}_1 - 1)\bar{a}_2}; -\bar{k}_1 |x(0)|^{\bar{a}_1 - 1}\right). \quad (10)$$

During the control design of FTSMs, there often exists a singularity problem. The reason is that the designed controllers have nonlinear terms $x^{-a}(a>0)$, which results in the unboundness of the control input when x diminishes to zero. To overcome this difficulty, NFTSM is put forward in the following.

Definition 3

NFTSM is described as

$$\sigma(t)=x+k_1'\,\mathrm{sign}^{a_1'}x+k_2'\,\mathrm{sign}^{a_2'}\dot{x}=0,\qquad k_1'>0,\ k_2'>0,\ 1<a_2'<2,\ a_1'>a_2'. \tag{11}$$

Remark 5

If $k'=k^{-}_2, k'=k^{-}_1a^{-}_2, a'=a^{-}_2, a'=a^{-}_1$, NFTSM (11) is equivalent to FTSM (7). It can be simply

verified in the following. It follows from NFTSM (11) that $\mathrm{sign}x=-\mathrm{sign}\dot{x}$ and $k_2'|\dot{x}|^{a_2'}=|x|+k_1'|x|^{a_1'}$. Owing to the relationship of the involved parameters, the latter equality can be rewritten as $|\dot{x}|=k^{-}_1(|x|+k^{-}_2|x|^{a^{-}_1})^{1/a^{-}_2}$. Also, it is apparent that $|x+k^{-}_2x^{a^{-}_1}|=|x|+k^{-}_2|x|^{a^{-}_1}$. Thus, we have

$$\dot{x}=|\dot{x}|\cdot\mathrm{sign}\dot{x}=-k^{-}_1(|x+k^{-}_2x^{a^{-}_1}|)^{1/a^{-}_2}\,\mathrm{sign}(x+k^{-}_2x^{a^{-}_1})=-k^{-}_1\,\mathrm{sign}^{1/a_2}(x+k^{-}_2x^{a^{-}_1}), \tag{12}$$

which implies the FTSM (7). Clearly, the sufficiency of (11) has been ensured. Indeed, the necessity of (11) can also be verified by using the similar process. In addition, it can be concluded that the settling time of the dynamics (11) is given by

$$T_{nf}=\int_0^{|x(0)|}\frac{{}'1/a_2'k\,2}{(x+k_1'x^{a_1'})^{1/a_2'}}\,\mathrm{d}x=\frac{a_2'|x(0)|^{1-1/a_2'}}{k_1'(a_2'-1)}$$

$$\cdot\mathrm{F}\left(\frac{1}{a_2'(a_1'-1)a_2'},\ a_2'-1;1+\frac{a_2'-1}{(a_1'-1)a_2'};-k'|x(0)|^{a_1'-1}\right). \tag{13}$$

Lemma 1 (*Jensens inequality* [*25, 26*])

$$\sum_{i=1}^{m}\theta_i^{\beta_2}\le\left(\sum_{i=1}^{m}\theta_i^{\beta_1}\right)^{\beta_2/\beta_1},\qquad 0<\beta_1<\beta_2 \tag{14}$$

with $\theta_i\ge0,\ 1\le i\le m$.

Proposition 1

Given the equalities

$$k'=(k'/k')^{a'/a_2'}_2{}_1,\ a'=a'/a',\quad k=k'^{-1/a_2'}_1{}_2,\quad k=(k'/k')^{1/a_2'}_2{}_1,\ a=a'^{-1}_1{}_2,\ a=a'a'^{-1}_2{}_1 \tag{15}$$

we can derive that

$$T_n>T_{nf}\ge T_{f\,1}, \tag{16}$$

i.e. for any nonzero initial state $x(0)$, the convergence time of NFTSM (11) is shorter than that of NTSM (2b), and not shorter than that of FTSM (3).

Proof

It follows from Lemma 1 that

$$(x+k_1' x^{a_1'})^{1/a_2'} \le x^{1/a_2'} + k_1'^{1/a_2'} x^{a_1'/a_2'}, \ x>0, \quad (17)$$

which implies

$$T_{nf} = \int_0^{|x(0)|} \frac{k_2'^{1/a_2'}}{(x+k_1' x^{a_1'})^{1/a_2'}} dx \ge \int_0^{|x(0)|} \frac{1}{k_2'^{-1/a_2'} x^{1/a_2'} + (k_2'/k_1')^{1/a_2'} x^{a_1'/a_2'}} dx = T_{f1}. \quad (18)$$

On the other hand, it is obvious that

$$(x+k_1' x^{a_1'})^{1/a_2'} > k_1'^{1/a_2'} x^{a_1'/a_2'}, \ x>0, \quad (19)$$

which follows that

$$T_{nf} = \frac{\int_0^{|x(0)|} k_1'^{1/a_2'} dx}{\int_0^{|x(0)|} (x+k_1' x^{a_1'})^{1/a_2'}} < \frac{k_1'^{1/a_2'}}{(k_2'/k_1')^{1/a_2'}} \int_0^{|x(0)|} x^{a_1'/a_2'} dx = T_n. \quad (20)$$

This completes the proof.

CONTROL DESIGN OF THE NFTSM CONCEPT

To briefly introduce the basic principle of NFTSM control technique, stabilization of a simple second-order nonlinear system is investigated in Section 3.1. It is concluded that the closed-loop system converges fast to the equilibrium within finite time while no singularity occurs. Then, in Section 3.2, the nonsingular technique is adopted to control a class of uncertain MIMO nonlinear systems. Accordingly, the proposed algorithm can be utilized to high-precision control of many mechanical systems, like robotic systems in practical applications.

A simple analysis
Consider the second-order dynamical system
$\dot{x}_1 = x_2,$
$\dot{x}_2 = f(x_1, x_2) + g(x_1, x_2)u,$ (21)
where x_1, x_2 are the system states, and $f(\cdot), g(\cdot)$ are nonlinear functions of x_1, x_2, and u is the control input.
According to the strategy of NFTSM (11), we define the following switching surface:
$\sigma = x_1 + a\,\mathrm{sign}^{\ }x_1 + \beta\,\mathrm{sign}^{\ }x_2,$ (22)
where $a>0$, $\beta>0$, $1<\ _2<2$, $_1>\ _2$. It is easy to see that the sliding surface (22) establishes the fast finite-time convergence. To make (21) reach the equilibrium fast along the given manifold, the control u is chosen as
$u = -1/g[1/(\beta\ _2)\,\mathrm{sign}^{2-\ _2}x_2 \cdot (1 + a\ _1|x_1|^{\ _1-1}) + f + K_1\,\mathrm{sign}\sigma + K_2\sigma],$ (23)
in which $K_1>0$, $K_2>0$. In the sequel, we will testify that the prescribed manifold (22) is reached fast within finite time.
Theorem 1
For the system (21), if the sliding surface is chosen as (22), and it is controlled by u expressed in (23), then the system trajectory will converge fast to zero within finite time. Moreover, no occurring of singularity is ensured during the whole process.
Proof
Consider the Lyapunov function candidate as
$V_1 = \frac{1}{2}\sigma^2,$ (24)
whose time derivative along (21) is
$\dot{V}_1 = \sigma \cdot (x_2 + a\ _1|x_1|^{\ _1-1}x_2 + \beta\ _2|x_2|^{\ _2-1}(f + g \cdot u)).$ (25) From (23), it follows that
$\dot{V}_1 = -K_1\beta\ _2|x_2|^{\ _2-1}|\sigma| - K_2\beta\ _2|x_2|^{\ _2-1}\sigma^2$
$= -\sqrt{2}^{\ _2-1}K_1\beta\ _2|x_2|^{\ _2-1}V_1^{1/2} - 2K_2\beta\ _2|x_2|^{\ _2-1}V_1,$ (26)
which can be rewritten as
$\dot{V}_1 = -\rho_1(x_2)V_1^{1/2} - \rho_2(x_2)V_1,$ (27)
where $\rho_1(x_2) = \sqrt{2}K_1\beta\ _2|x_2|^{\ _2-1}$ and $\rho_2(x_2) = 2K_2\beta\ _2|x_2|^{\ _2-1}$. It can be derived that when $x_2 \neq 0$, $\rho_1>0$, $\rho_2>0$ and thus (27) has the form of FTSM (3) since $V_1\geq 0$. In the light of the analysis in Section 2, it is concluded that for the case $x_2 \neq 0$, the system states will move fast to the sliding

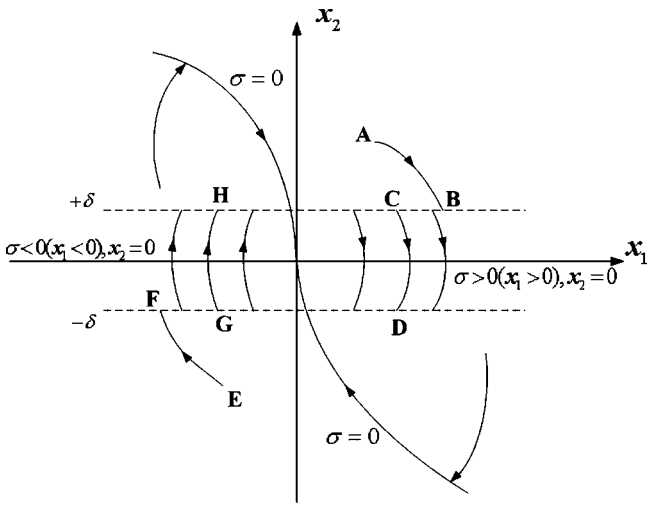


Figure 2. The phase-trajectory analysis of the system.

mode $\sigma = 0$ within finite time. Then we consider the case $x_2 = 0$. By substituting (23) into the second equation of (21), one can obtain

$$\dot{x}_2 = -1/\beta \operatorname{sign}^{2-\gamma_2} x_2 \cdot (1+a|x_1|^{\gamma_1-1}) - K_1 \operatorname{sign}\sigma - K_2\sigma. \quad (28)$$

From $x_2 = 0$, (28) can be rewritten as

$$\dot{x}_2 = -K_1 \operatorname{sign}\sigma - K_2\sigma, \quad (29)$$

which suggests that $\dot{x}_2 < -K_1$ and $\dot{x}_2 > K_1$ for the cases $\sigma>0$ and $\sigma<0$, respectively. It is easy to understand that the system will not always stay on the points ($x_1 = 0$ and $x_2 = 0$). Moreover, it is reasonable to assume that there exists a vicinity of $x_2 = 0$, i.e. $|x_2| \le \partial$ (∂ is a small positive constant), satisfying $\dot{x}_2 < -K_1$ and $\dot{x}_2 > K_1$ for $\sigma>0$ and $\sigma<0$, respectively. Thus, it follows that the crossing of trajectories between two boundaries of $|x_2| \le \partial$ is performed in finite time, and the dynamics from the region $|x_2| > \partial$ converge to the boundaries in finite time as well [16]. The analysis of phase trajectory is presented in Figure 2. Therefore, it is concluded that the sliding mode $\sigma = 0$ can be reached from anywhere in the phase plane in finite time. Besides, in comparison with the form $\dot{x}_2 = -K \operatorname{sign}\sigma (K>0)$ in [16], (29) ensures that the crossing of trajectories between $x_2 = \partial$ and $x_2 = -\partial$, and the convergence from the region $|x_2| > \partial$ to $\sigma = 0$ occur at a faster rate due to the additional linear term $-K_2\sigma$.

It is noted that the control (23) does not contain any singularity since $\gamma_1 > 1$ and $\gamma_2 < 2$. Hence, we conclude that from any initial states, the closed-loop system may converge quickly to the origin along NFTSM (11) in finite time without any singularity. This completes the proof.

Remark 6

From above, we can get the conclusion that u (23) is designed to achieve

$$\dot{\sigma} = -\rho_1^*(x_2)\operatorname{sign}\sigma - \rho_2^*(x_2)\sigma, \quad \rho_1^* = \rho_1/\sqrt{2} \ge 0,\ \rho_2^* = \rho_2/2 \ge 0. \quad (30)$$

It is clear to see that (30) coincides with the constant plus proportional reaching law in [2], provided that $x_2 = 0$ which often holds before $\sigma = 0$ is arrived. As is known, such a reaching law can accelerate the convergence rate toward the manifold due to the proportional term $\rho_2^*\sigma$, and thus it enhances the property of robustness. Moreover, it is concluded that by using this control

law, the chattering phenomenon can be alleviated when system () operates on the sliding surface.
However, in essence, chattering is caused by the discontinuity of the sign function. In order to totally eliminate the discontinuity, we propose

$$\psi(\sigma) = \frac{e^{b\sigma} - 1}{e^{b\sigma} + 1}, \quad b>0, \qquad (31)$$

to replace the sign function in (23). It can be shown that the larger the parameter b, the better performance we will get. One can choose a suitable parameter in practical implementation.

Application of practical systems
Consider the uncertain nonlinear system

$$\begin{aligned} \dot{X}_1 &= F_1(X_1, X_2), \\ \dot{X}_2 &= F_2(X_1, X_2) + H(X_1, X_2) + B(X_1, X_2)U, \end{aligned} \qquad (32)$$

where the states $X_1 = [x_{11}, x_{12}, \ldots, x_{1n}] \in R^n$, $X_2 = [x_{21}, x_{22}, \ldots, x_{2n}] \in R^n$, $F_1(\cdot), F_2(\cdot)$ are nonlinear smooth vector functions, $H(\cdot)$ represents the uncertainties and disturbances, $B(\cdot)$ is a nonsingular matrix and $u \in R^n$ is the control vector. For the convenience of study, we give the following assumptions.
Assumption 1
$[X_1, X_2]=0$ if and only if $[X_1, \dot{X}_1]=0$.
Assumption 2
$H(X_1, X_2)$ is a bounded function satisfying

$$\| H(X_1, X_2) \| \le \rho(X_1, X_2), \qquad (33)$$

where $\rho(X_1, X_2)$ denotes the upper bound of the system uncertainty..
Assumption 3
The matrix $(bF_1/bX_2)B(X_1, X_2)$ is nonsingular.
It should be pointed out that plenty of practical dynamical systems, especially the mechanical systems, can be expressed in (32) satisfying the above conditions. For example, robotic systems [8, 16, 19, 21, 22] are not exactly in the form of (32), but it can be transformed into such a form by the coordinates' change. Therefore, the proposed algorithm in the work can be applied to any plant, which can be transformed to (32).
Now, we construct the NFTSM for the system (32) as

$$\Sigma = X_1 + \Omega_1 \mathrm{sign}^{F1} X_1 + \Omega_2 \mathrm{sign}^{F2} \dot{X}_1, \qquad (34)$$

in which Σ denotes n-dimensional switching manifold, and the involved matrices are represented by

$$\Omega_1 = \mathrm{diag}(\beta_{11}, \beta_{12}, \ldots, \beta_{1n}), \quad \Omega_2 = \mathrm{diag}(\beta_{21}, \beta_{22}, \ldots, \beta_{2n}),$$
$$F_1 = \mathrm{diag}(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}), \quad F_2 = \mathrm{diag}(\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n}),$$

with $\beta_{1i}>0, \beta_{2i}>0$, $\alpha_{1i} > \alpha_{2i}, 1 < \alpha_{2i} < 2$, for every $i = 1, 2, \ldots, n$. Besides, $\mathrm{sign}^{F1} X_1$ is a vector defined as

$$\mathrm{sign}^{F1} X_1 = \mathrm{diag}(\mathrm{sign} X_1) \cdot |X_1|^{F1} = [\mathrm{sign}^{\alpha_{11}} x_{11}, \mathrm{sign}^{\alpha_{12}} x_{12}, \ldots, \mathrm{sign}^{\alpha_{1n}} x_{1n}]^T$$

and $\mathrm{sign}^{F2} \dot{X}_1$ can also be defined accordingly.

To facilitate the control design, we adopt the notion that

$$\frac{d}{dt}(\operatorname{sign}(X_1)|X_1|^{F_1}) = F_1 \operatorname{diag}(|X_1|^{F_1 - I_n}) \cdot X_1, \quad \frac{d}{dt}(\operatorname{sign}(X_1)|X_1|^{F_2}) = F_2 \operatorname{diag}(|X_1|^{F_2 - I_n}) \cdot X_1, \quad (35)$$

where I_n is the n-dimensional unit matrix, and it can be easily verified.
In what follows, we will present a criterion for fast finite-time control of system (32).
Theorem 2
Under Assumption 1–3, the following control law:

$$U=-\left(\frac{\partial \dot{F}_1}{\partial X_2}B\right)^{-1}\left(M_2\Sigma+\left(\frac{\partial \dot{F}_1}{\partial X_2}\right)^{\cdot}\rho+M_1\frac{\Sigma}{\|\Sigma\|}+\frac{\partial F_1}{\partial X_1}\dot{F}_1+\frac{\partial F_1}{\partial X_2}\dot{F}_2\right)$$
$$+\Omega^{-1}F^{-1}(I_n+\Omega_1F_1\operatorname{diag}(|X_1|^{F1-In}))\operatorname{sign}^{2In-F2}(\dot{X}_1),\quad M_1>0,\ M_2>0, \quad (36)$$

along with the switching manifold (34) is capable of globally regulating the uncertain system (32) at the origin within finite time, while no singularity occurs.
Proof
The Lyapunov function is chosen as

$$V_2=\frac{1}{2}\Sigma^T\Sigma. \quad (37)$$

From (35), its time derivative can be given by

$$\dot{V}=\Sigma^T\dot{\Sigma}=\Sigma_2^T\cdot[\dot{X}_1+\Omega_1F_1\operatorname{diag}(|X_1|^{F1-In})\cdot\dot{X}_1+\Omega_1F_1\operatorname{diag}(|\dot{X}_1|^{F2-In})\cdot\ddot{X}_1], \quad (38)$$

Since

$$\ddot{X}_1=\frac{d}{dt}F_1(X_1,\dot{X}_1)=\frac{\partial \dot{F}_1}{\partial X_1}\dot{X}_1+\frac{\partial \dot{F}_1}{\partial \dot{X}_1}\ddot{X}_1=\frac{\partial \dot{F}_1}{\partial X_1}\dot{X}_1+\frac{\partial \dot{F}_1}{\partial \dot{X}_1}(F_2+H+BU), \quad (39)$$

it can be obtained that

$$\dot{V}=\Sigma_2^T\dot{X}_1+\Omega_1F_1\operatorname{diag}(|X_1|^{F1-In})\dot{X}_1+\Omega_2F_2\operatorname{diag}(|\dot{X}_1|^{F2-In})\frac{\partial \dot{F}_1}{\partial X_1}\dot{X}_1+\frac{\partial \dot{F}_1}{\partial \dot{X}_1}(F_2+H+BU). \quad (40)$$

Substituting (36) into (40), it follows that

$$\dot{V}=-\Sigma_2^T\cdot\Omega_2F_2\cdot\operatorname{diag}(|\dot{X}_1|^{F2-In})\cdot\left(M_1\frac{\Sigma}{\|\Sigma\|}+M_2\Sigma+\frac{\partial \dot{F}_1}{\partial X_1}\dot{X}_1\rho\frac{\Sigma}{\|\Sigma\|}-\frac{\partial \dot{F}_1}{\partial X_1}H-\frac{\partial \dot{F}_1}{\partial \dot{X}_1}\ddot{X}_1\right)$$

Define $\leq \min_{i=1,\ldots,n}(\beta_{2i}|\dot{x}_i|^{2-1})\cdot(M_1\|\Sigma\|+M_2\Sigma^T\Sigma). \quad (41)$

$$\lambda_1(\dot{X}_1)\min_{i=1,\ldots,n}=(\beta_{2i}|\dot{x}_i|^{2-1})-\sqrt{2}M_1,\ \lambda_2(\dot{X}_1)=\min_{i=1,\ldots,n}(\beta_{2i}|\dot{x}_i|^{2-1})\cdot 2M_2. \quad (42)$$

Equation (41) can be rewritten as

1/2

$\dot{V}_2 \leq - V - V_2$. (43)

By using Comparison Lemma [27] and the similar arguments in the proof of Theorem 1, it can be concluded that the fast finite-time convergence to $\Sigma = 0$ is established, and thus the uncertain MIMO systems will converge to $X_1 = X_2 = 0$ at a fast rate within finite time. Besides, one can easily see that the NTSM control (36) does not involve any terms that have negative powers.

Consequently, it is claimed that the NTSM control (36) provides the closed-loop system with fast finite-time convergence rate and singularity avoidance during the whole control process. This completes the proof.

Remark 7

It can be found that the control (36) is designed to satisfy

Σ

$\dot{\Sigma} = - \alpha_1^*(x_2) - \alpha_2^*(x_2)\Sigma, \quad (44) \quad \overline{\|\Sigma\|}$

$F_2 - I_n$

in which $\alpha_1^* = M_1\Omega_2 F_2 \cdot \mathrm{diag}(|\dot{X}_1|) + \| bF_1/bX_2 \cdot \rho \|(bF_1/bX_2)(H \cdot \Sigma / \|\Sigma\|) \geq 0$, $\alpha_2^* =$

F

$M_2\Omega_2 F_2 \cdot \mathrm{diag}(|\dot{X}_1|^{2-I_n}) \geq 0$. It is obvious that (44) can be regarded as the multidimensional

form of (31). Thus accordingly, such a reaching law may provide the closed-loop system with

strong robustness and reduce the possible chattering phenomenon caused by imperfect switching. Likewise, in order to further get rid of the discontinuous property, the multidimensional switching function $\Sigma / \|\Sigma\|$ can be substituted by

$$T(\sigma) = \frac{\Sigma}{\|\Sigma\| + \partial'} \quad (45)$$

where $\partial > 0$ is a small constant.

NUMERICAL SIMULATION

To illustrate the effectiveness and applicability of the proposed control algorithm, we design the NFTSM controller for high-precision tracking of a two-link rigid robot manipulator, whose simulation results are compared with the NTSM control scheme.

Consider that the dynamic equation of the manipulator model [16] is given by

$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau(t) + \tau_d$, (46) where $q, \dot{q}, \ddot{q} \in R^2$ are the vectors of joint angular position, velocity and acceleration, respectively.

$M(q)$ is the 2×2 symmetric positive-definite inertia matrix, $C(q, \dot{q})$ is the 2×1 vector containing

Coriolis and centrifugal forces, $G(q)$ is the 2×1 gravitational torque and $\tau(t)$ is 2×1 vector of applied joint torques that are actually the control inputs and τ_d denotes the 2×1 bounded input disturbance vector. Moreover, the involved matrices can be expressed as

$$M = \begin{matrix} (m_1 + m_2)r^2 + m_2 r^2 + 2m_2 r_1 r_2 \cos q_2 + J_1 & m_2 r^2 + m_2 r_1 r_2 \cos q_2 \\ m_2 r^2 + m_2 r_1 r_2 \cos q_2 & m_2 r^2 + J_2 \end{matrix}$$

$$C = \begin{matrix} -\beta_{12}(q_2)\dot{q}_1^2 - 2\beta_{12}(q_2)\dot{q}_1\dot{q}_2 \end{matrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} \tau_1 \\ \beta_{12}(q)\dot{q}_2^2 \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_{d1} \\ \tau_{d2} \end{bmatrix}, \quad \tau_d = \begin{bmatrix} \tau_2 \end{bmatrix},$$

$=$

$(m_1 + m_2)gr_1 \cos q_2 + m_2 gr_2 \cos(q_1 + q_2)$

$G(q)$

$m_2 gr_2 \cos(q_1 + q_2)$

with parameters given as r_1 1 m, r_2 0.8 m, J_1 5 kgm, J_2 5 kgm, m_1 0.5 kg, m_2 1.5 kg. It is assumed that rigid robotic manipulators have uncertainties, i.e.

$M(q) = M_0(q) + \partial M(q), \quad C(q;\dot{q}) = C_0(q;\dot{q}) + \partial C(q;\dot{q}) \quad G(q) = G_0(q) + \partial G(q), \quad (47)$

where $M_0(q)$, $C_0(q;\dot{q})$, $G_0(q)$ are the estimated terms, and $\partial M(q), \partial C(q;\dot{q}), \partial G(q)$ are uncertain terms. Here we assume that the nominal values of m_1 and m_2 are $m_{10} = 0.4$ kg, $m_{20} = 1.2$ kg, and

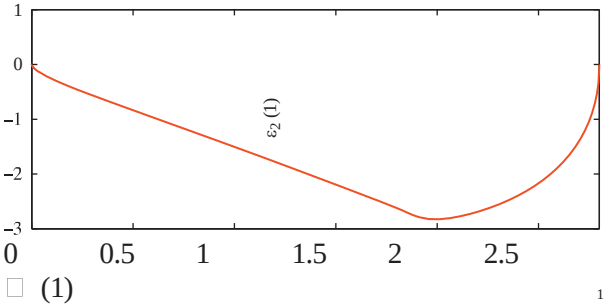
the external disturbances are $\tau_d = [2\sin(t) + 0.5\sin(200\pi t), \cos(2t) + 0.5\sin(200\pi t)]^T$. The initial

values of the system are selected as $q_1(0) = 3.0, q_2(0) = 2.5, \dot{q}_1(0) = 0$ and $\dot{q}_2(0) = 0$, and the desired reference signals are given by $q_d = [q_{d1}, q_{d2}]^T$ with $q_{d1} = 1.25 - 7/5\exp(-t) + 7/20\exp(-4t)$ and

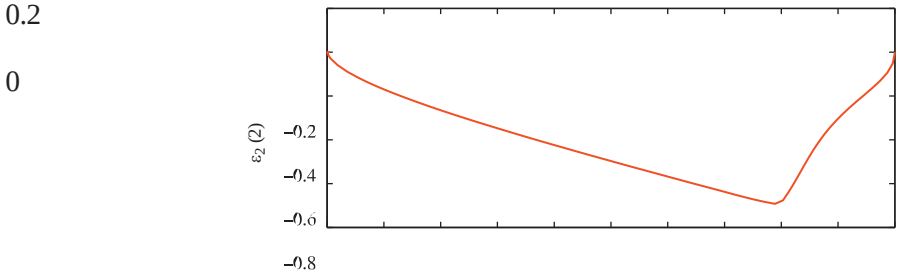
$q_{d2} = 1.25 + \exp(-t) - (1/4)\exp(-4t)$.

Define $\varepsilon_1(t) = q - q_d$ and $\varepsilon_2(t) = \dot{q} - \dot{q}_d$. The error equation of the rigid robotic manipulator can be obtained as follows:

$$\dot{\varepsilon}_1 = \varepsilon_2, \qquad (48)$$
$$\dot{\varepsilon}_2 = F_2(\varepsilon_1, \varepsilon_2) + H(\varepsilon_1, \varepsilon_2) + B(\varepsilon_1, \varepsilon_2)\tau,$$



(1)



0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5
(2)
1
Figure 3. The phase plot of tracking error of joints 1 and 2.

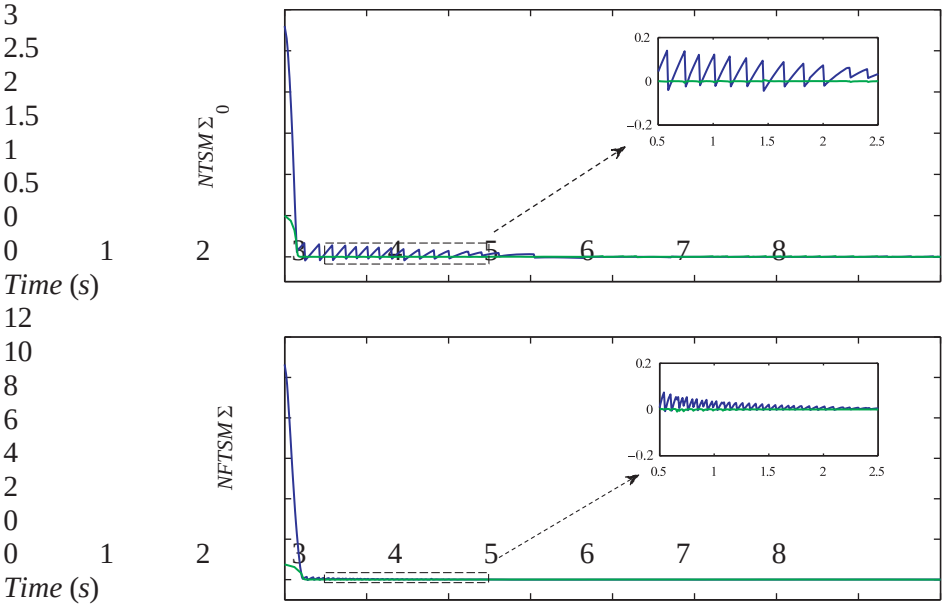


Figure 4. The time evolution of NTSM Σ and NFTSM Σ_0.

in which $F_2(\varepsilon_1,\varepsilon_2)=-M^{-1}C_0-M^{-1}G_0\ddot{q}_d$, $H(\varepsilon_1,\varepsilon_2)=-M^{-1}(\partial M_0\ddot{q}+\partial C+\partial G-\tau_d)$ and $B(\varepsilon_1,\varepsilon_2)=M^{-1}$. The uncertainty term $H(\varepsilon_1,\varepsilon_2)$ is assumed to satisfy the following:

$$\|H(\varepsilon_1,\varepsilon_2)\|\le\|M^{-1}\|\cdot(b_0+b_1\|q\|+b_2\|\dot{q}\|^2), \quad (49)$$

where $b_0=12, b_1=2.2, b_2=2.8$. It can be found that the error system will have the form of (32),
and the upper-bounded function is $\rho(\varepsilon_1,\varepsilon_2)=\|M(0)^{-1}\|(b_0+b_1\|q\|+b_2\|\dot{q}\|)$.
According to Theorem 2, we define the sliding surface as

$$\Sigma=\varepsilon_1+F_1\varepsilon_1+F_2\varepsilon_2 \quad (50)$$

sign ϵ sign ϵ ,

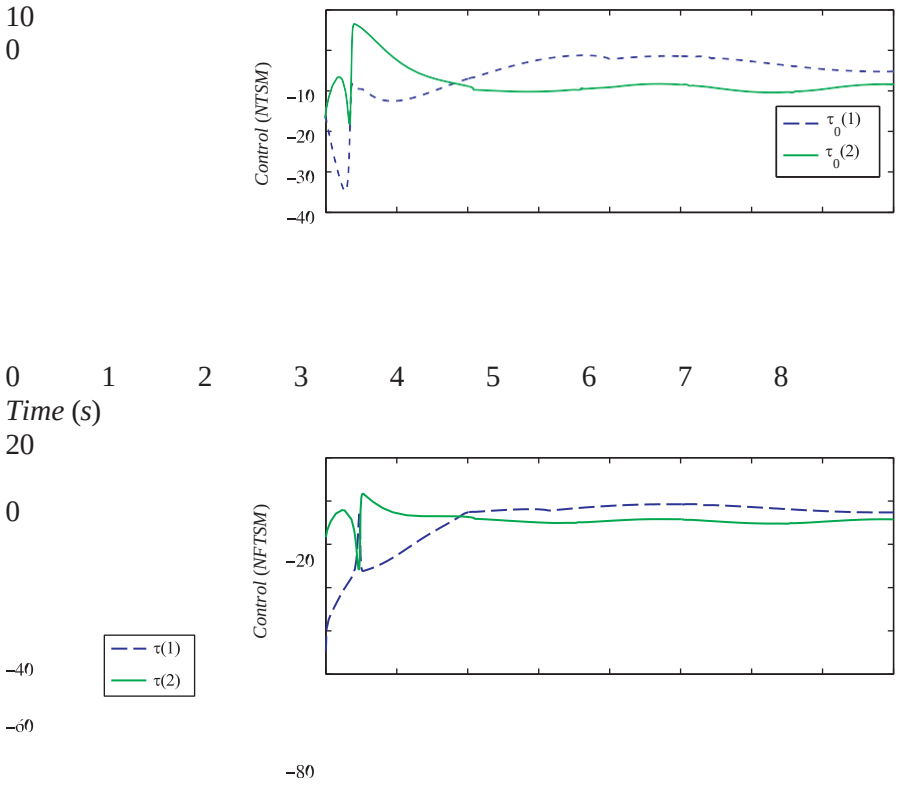
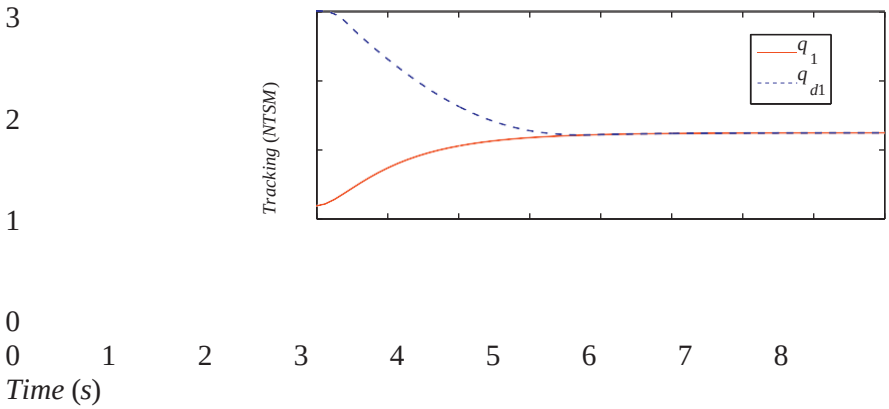


Figure 5. The time evolution of control signals τ_0 and τ.



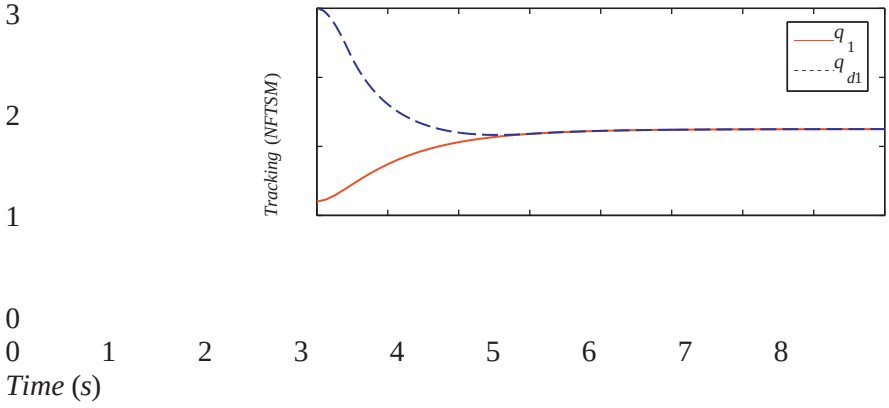


Figure 6. Output tracking of joint 1 in the case of NTSM and NFTSM.

with

$$F_1 = \begin{matrix} 2 & \\ & 5 \end{matrix}, \quad F_2 = \begin{matrix} 0 & \\ 0 & \end{matrix}.$$

Then, the following controller is designed:

$$\tau = -M_0[M_2\Sigma + (\rho + M_1) + F_2 + F^{-1}(I_2 + F_1 \operatorname{diag}(|\varepsilon_1|^{\alpha_2}))\operatorname{sign}(\varepsilon_2)], \quad (51)$$

in which $M_1 = 1$, $M_2 = 2$.

For comparison purposes, we design the following sliding surface and NTSM controller in the light of Theorem 2 [16] as

$$\Sigma = \varepsilon_2 + \frac{1}{\beta_0}\operatorname{sign}^{\alpha_1}(\varepsilon_2), \quad \tau = -M_{10}[(\rho + M_{10})\Sigma + F_2 + F^{-1}\operatorname{sign}^{2-\alpha_2}(\varepsilon_2)], \quad (52)$$

where $M_{10} = 4.3$ and

$$F = \begin{matrix} 5 & \\ & 10 \end{matrix}.$$

5 0
3
.
The simulation plots are presented in the following to demonstrate the superior performance of
faster and high-precision tracking with the proposed NFTSM scheme. The sample time is given as t_s 0.01 s. The phase plots of tracking errors of NFTSM system's joints 1 and 2 are reported in Figure 3. It can be seen from Figure 4 that the tracking errors reach NTSM Σ_0 0 and NFTSM Σ 0 within the finite time. Moreover, in comparison with the NTSM case, less chattering occurs during the NFTSM control process, which is by reason of a smaller gain for the switching term
in (51).
In order to further eliminate the chattering, the approximated function (45) is adopted with
∂ 0.01. Figure 5 depicts the control signals of two joints. It is noticed that neither singularity nor chattering occurs during the whole NFTSM control process. As shown in Figures 6 and 7, the system states track the desired reference signals quickly in the cases of NTSM and NFTSM,
respectively. Moreover, since the concept of NFTSM and the constant plus proportional reaching

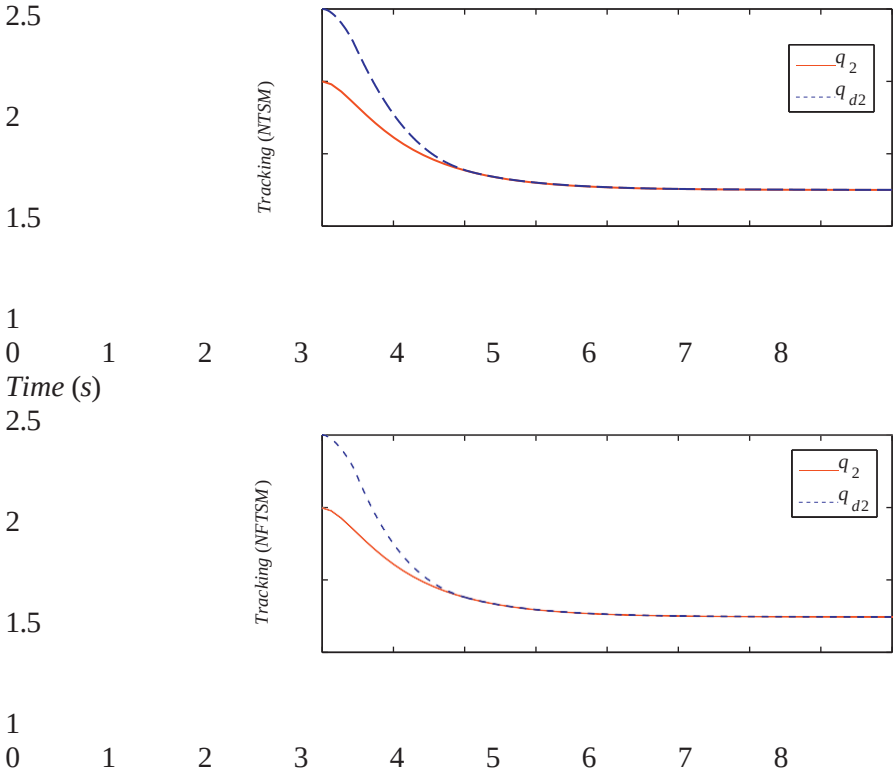


Figure 7. Output tracking of joint 2 in the case of NTSM and NFTSM.
law decides a faster reaching, it is seen that the NFTSM system is steered to reference signals within a shorter time.

CONCLUSIONS

We have examined high-precision control of nonlinear dynamics in this article. In order to demonstrate great resilience and quick finite-time convergence, many novel FTSM variants have been put forward. New FTSMs have been shown to perform better than the traditional one, particularly when the starting state is far from the equilibrium. The idea of NFTSM has been developed to prevent singularity during the control process thanks to the advantages of FTSM. A class of uncertain MIMO nonlinear dynamics, including mechanical systems, has been controlled using the nonsingular technique. The enhanced performance of the NFTSM control approach has been shown by numerical simulation on robotic manipulators. It should be mentioned that a variety of applications, including filter designs and chaotic suppression and synchronization, might easily include the suggested methodologies.

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